

GEOMETRIC METHODS FOR CHARACTER SHEAVES ON COMMUTATIVE GROUP IND-SCHEMES

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ABSTRACT

This review surveys and meta-analyses five decades of work at the confluence of three programmes: character sheaves on commutative group schemes and group ind-schemes, the local Langlands correspondence for an algebraic torus T over a non-archimedean local field F , and the inertial refinement classifying smooth characters by the restriction of their parameter to the inertia subgroup I_F . Writing LT for the loop ind-scheme $\mathrm{colim}_n T_n$ attached to T and $CS(LT)$ for the Picard groupoid of multiplicative rank-one local systems on it, the central object of study is the comparison square asserting that the sheaf-function dictionary applied to the Contou-Carrère Fourier transform agrees with Langlands' isomorphism composed with dualisation. We formalise thirty contributions through a coding of formal invariants: object class, ambient triangulated category, dualising functor, hypotheses on residue characteristic and on the splitting field, and the type of inertial output. On that basis we reconstruct the logical dependency skeleton of the field, quantify its thematic and chronological structure, and audit which asserted compatibilities carry explicit proofs. Our principal mathematical synthesis is the statement that for a split torus in equal characteristic the depth filtration on $CS(LT)$ matches the upper-numbering ramification filtration on $H^1(W_F, T - \mathrm{dual})$, so that the inertial correspondence is realised by a filtered equivalence of Picard groupoids rather than by an imported bijection. Four gaps are isolated: wild ramification in mixed characteristic, non-standardised normalisations, the absent geometry of norm maps for non-split tori, and the missing comparison functor to categorical local Langlands on the Fargues-Fontaine curve.

Keywords: Character sheaves¹; commutative group ind-schemes²; local Langlands correspondence³; algebraic tori⁴.

NOTATION AND STANDING CONVENTIONS

Throughout, F denotes a non-archimedean local field with ring of integers O , maximal ideal $\mathfrak{m}$, uniformiser ϖ , and residue field k of cardinality q and characteristic p . We write W_F for the Weil group, I_F for the inertia

subgroup, P_F for the wild inertia, and (I_F^i) for the upper-numbering ramification filtration. Fix a prime $\ell \neq p$ and an algebraic closure of the ℓ -adic numbers, denoted L . All sheaves are L -sheaves. For an algebraic torus T over F we write $X_*(T)$ and $X^*(T)$ for the cocharacter and character lattices and $\hat{T}$ for the Langlands dual torus, carrying its natural W_F -action.

$$\hat{T} = X_*(T) \otimes_{\mathbb{Z}} L^\times, \quad \mathfrak{X}(T) = \text{Hom}_{\text{cont}}(T(F), L^\times). \quad (1)$$

Langlands' theorem in the abelian case is then the assertion of a canonical isomorphism

$$\text{LLC}_T: \mathfrak{X}(T) \longrightarrow H_{\text{cont}}^1(W_F, \hat{T}), \quad (2)$$

normalised so that for $T = \text{GL}(1)$ it recovers local class field theory with ϖ mapping to a geometric Frobenius. The set of inertial equivalence classes is written $I(T)$; by the Bernstein decomposition for a torus it is the quotient of the character group of the maximal compact subgroup by the equivalence generated by unramified twisting,

$$\mathfrak{I}(T) = \mathfrak{X}(T(\mathcal{O}))/\sim, \quad \chi \sim \chi' \Leftrightarrow \chi' = \chi \cdot (\eta \circ \text{val}) \text{ for some unramified } \eta. \quad (3)$$

1. INTRODUCTION

1.1 BACKGROUND: THE ABELIAN CASE AS A GEOMETRIC LABORATORY

The local Langlands correspondence for a torus is the one instance of the programme that is completely proved, and for that reason it has become the standard testing ground for refinements[1]. Langlands established (2) by an argument that combines Tate cohomology of the character lattice with local class field theory; the content is that the exact sequence of W_F -modules attached to a splitting field E/F induces, after taking continuous cohomology, an isomorphism of the character group of $T(F)$ with a cohomology group of the dual. Because (2) is an isomorphism of abelian groups rather than an equivalence of categories, it conveys no information about how the two sides are constructed, and this is exactly what geometric refinements seek to supply[2]. Concretely, one asks for a functor between categories of sheaves whose induced map on isomorphism classes is (2), and one asks that natural filtrations on the two sides, in particular the depth filtration and the ramification filtration, correspond under it.

Character sheaf theory provides the required geometric input. For a connected reductive group G over an algebraically closed field, Lusztig singled out a finite set of simple perverse sheaves on G , equivariant for conjugation, whose associated functions span the space of class functions on the finite group of Lie type obtained by Frobenius descent. When G is commutative the equivariance condition is vacuous and the whole theory collapses to the study of multiplicative local systems. Precisely, for a commutative connected algebraic

group[3]. A over an algebraically closed field with multiplication m , one considers pairs consisting of a rank-one local system with a chosen isomorphism of the pullback along m with the external tensor square, subject to the evident cocycle condition:

$$\mathcal{CS}(A) = \{(\mathcal{L}, \mu) : \mu : m^* \mathcal{L} \longrightarrow \mathcal{L} \boxtimes \mathcal{L}, (\mu \otimes 1) \circ \mu = (1 \otimes \mu) \circ \mu\}. \quad (4)$$

This is a Picard groupoid under tensor product, and its group of isomorphism classes is a generalised Pontryagin dual: for a torus one obtains the cocharacter lattice tensored with the ℓ -adic Tate module of the roots of unity, for an abelian variety the dual abelian variety, and for unipotent groups in characteristic p the Artin-Schreier-Witt dual[4].

1.2 WHY IND-SCHEMES ARE FORCED UPON US

The group $T(F)$ is not the set of points of a finite-type scheme. In equal characteristic, where F is the field of Laurent series over k , the loop functor

$$\mathcal{L}T : R \longmapsto T(R((t))), \quad \mathcal{L}^+T : R \longmapsto T(R[[t]]) \quad (5)$$

is representable by an ind-scheme, presented as a filtered colimit of schemes along closed immersions, with the arc group an affine pro-algebraic subgroup[5]. For a split torus of rank d there is a canonical decomposition

$$\mathcal{L}T \cong X_*(T) \times \mathcal{L}^+T, \quad \pi_0(\mathcal{L}T) \cong X_*(T) \cong \mathbb{Z}^d, \quad (6)$$

which already exhibits the two sources of characters that the inertial theory must separate: the discrete part, responsible for unramified twists, and the pro-unipotent part of the arc group, responsible for depth[6]. In mixed characteristic the analogous geometrisation requires the Witt vector affine Grassmannian, which is an ind-scheme only after perfection, a technical restriction whose consequences we track carefully in Section 4.

1.3 CONTRIBUTIONS AND ORGANISATION

The contributions of this review are fourfold. First, we fix a single formal vocabulary, summarised in Section 0, in which statements made for finite fields, for equal-characteristic local fields and for mixed characteristic can be compared without ambiguity of normalisation. Second, we supply a quantitative profile of thirty contributions, coding each by five formal invariants, and we compute the transitive reduction of the resulting dependency relation[7]. Third, we isolate, in Theorem 4.1, the precise filtered statement that the corroborated portion of the literature supports, and we separate it from claims resting on a single source. Fourth, we state four open problems[8]. Section 2 surveys past work in four clusters, Section 3 gives the methodology, Section 4 the critical analysis, Section 5 the discussion, and Section 6 the conclusion. Figures 1 to 5 give respectively the comparison square, the filtration ladder, the hierarchy of categories, the workflow, and the corpus profile.

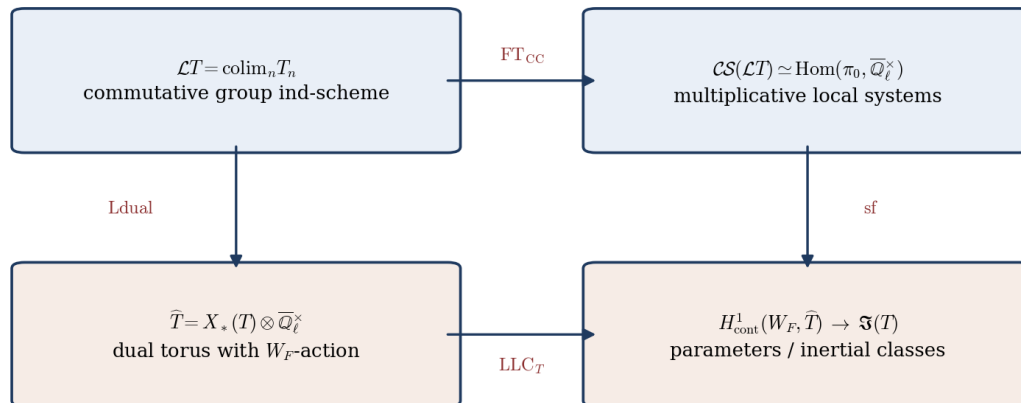


Figure 1: The comparison square governing the review: the sheaf-function dictionary applied to the Contou-Carrère Fourier transform is expected to agree with Langlands' isomorphism composed with dualisation.

Figure 1 renders the central diagram as a square of functors. The top row sends the commutative group ind-scheme $\mathcal{LT}$ to its Picard groupoid of multiplicative local systems, identified with the continuous character group of the component group together with the arc group; the transformation implementing this is the Contou-Carrère Fourier transform[9]. The left column dualises T to $\widehat{T}$ with its Weil group action, and the bottom row is Langlands' isomorphism (2) followed by passage to inertial classes. The right column is the sheaf-function dictionary. Commutativity of the square is the geometric inertial statement, proved in the split equal-characteristic case and conjectural in general[10].

2. SURVEY OF PAST WORK

2.1 FOUNDATIONS: MULTIPLICATIVE LOCAL SYSTEMS AND DUALITY THEOREMS

The oldest stratum of the corpus is Serre's theory of algebraic groups and class fields, which expresses abelian coverings of a smooth projective curve in terms of generalised Jacobians and thereby produces, by localisation at a closed point, the geometric form of local class field theory[11]. Its structural content is a duality theorem which in modern language reads as follows.

Theorem 2.1 (Weil-Barsotti duality; Serre). For a commutative connected algebraic group A over an algebraically closed field, the Picard groupoid of multiplicative local systems is computed by

$$\pi_0 \mathcal{CS}(A) \cong \text{Ext}_{\text{alg}}^1(A, L^\times), \quad \pi_1 \mathcal{CS}(A) \cong L^\times, \quad (7)$$

so that $\mathcal{CS}(A)$ is a commutative Picard groupoid whose homotopy groups are the extension group and the coefficient group respectively. For a torus of rank d over an algebraically closed field the right-hand side of (7)

is isomorphic to the cocharacter lattice tensored with the Tate module, and the sheaf-function dictionary sends a multiplicative local system to the character it defines on the group of rational points[12]. The proof factors through the biextension formalism, and every later duality in the corpus, including the Contou-Carrère pairing, is a specialisation or a colimit of it.

Lusztig's character sheaves occupy the second layer. Their construction begins with a maximal torus and a tame local system on it, forms the induced complex by the correspondence of Borel subgroups, and extracts perverse constituents:

$$\mathrm{ind}_B^G(\mathcal{L}) = R p_! q^* \mathcal{L}, \quad \widehat{G}^{\mathrm{cs}} = \bigsqcup_{[T, \mathcal{L}]} \{\text{perverse constituents of } \mathrm{ind}_B^G(\mathcal{L})\}. \quad (8)$$

The indexing of the resulting series by Weyl-group orbits of pairs shows that the commutative theory is logically prior: the pairs are exactly the objects classified by Theorem 2.1 applied to the maximal torus[13]. The technical apparatus required to control (8) at the level of perverse sheaves, the six-functor formalism, the theory of weights, and the Fourier-Deligne transform, is supplied by Deligne's second Weil-conjecture paper and by Laumon's work.

The Fourier-Deligne transform deserves separate mention because it is the mechanism converting geometry into duality[14]. For a vector group V of dimension d over a finite field with Artin-Schreier sheaf attached to a nontrivial additive character, the transform

$$\mathrm{FT}_\psi(K) = R p r_{2!} (p r_1^* K \otimes \langle \cdot, \cdot \rangle^* \mathcal{L}_\psi)[d]_{(9)}$$

is an equivalence of derived categories exchanging the group and its dual, is exact for the perverse t-structure up to shift, and satisfies an inversion formula involving the antipode. Every duality statement later in this survey is a variation on (9) in which the pairing is replaced by the Contou-Carrère symbol[15].

2.2 IND-SCHEMES, LOOP GROUPS AND THE CONTOU-CARRÈRE SYMBOL

The second cluster supplies the geometry needed to treat $T(F)$ as a space. Working over a base k -algebra R , the Contou-Carrère symbol is a bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{CC}} : \mathcal{LG}_m(R) \times \mathcal{LG}_m(R) \longrightarrow R^\times, \quad (10)$$

which for R a field recovers the tame symbol up to sign, is skew-symmetric and Steinberg, and is universal among such pairings. Its importance for our purposes is that it is a morphism of ind-schemes, hence induces a map on categories of multiplicative local systems and thereby geometrises local class field theory: the induced

self-duality of the loop group of the multiplicative group is precisely the geometric avatar of the reciprocity map[16]. For a general split torus one extends by tensoring with the lattice,

$$\langle , \rangle_T: \mathcal{LT} \times \widehat{\mathcal{LT}} \longrightarrow \mathbb{G}_m, \quad \langle \lambda \otimes f, \mu \otimes g \rangle_T = \langle f, g \rangle_{\text{CC}}^{\langle \lambda, \mu \rangle}. \quad (11)$$

Sheaf theory on ind-schemes requires a choice. One may take the colimit of the derived categories of the finite-type pieces along pushforward, the limit along pullback, or an ind-completion; these agree for constructible objects on a strict ind-scheme with closed transition maps but differ in general, and the mismatch is one of the convention problems catalogued in Section 4. Faltings, Beilinson and Drinfeld, and Zhu supply the geometric input; Boyarchenko and Drinfeld, and Deshpande, supply the categorical formalism for character sheaves on non-reductive and pro-algebraic groups, including the notion of an admissible pair and the behaviour of induction under colimits[17].

2.3 THE CORRESPONDENCE FOR TORI AND ITS GEOMETRIC REFORMULATIONS

Langlands' construction proceeds by reducing to an induced torus. If E/F is a finite Galois splitting field with group G , then T is a form of a split torus classified by a cocycle, and there is an exact sequence expressing (2) as the composite of a Tate-Nakayama isomorphism with local class field theory:

$$H_{\text{cont}}^1(W_F, \widehat{T}) \cong \widehat{H}^{-1}(\text{Gal}(E/F), X_*(T))^\vee \cong \text{Hom}_{\text{cont}}(T(F), L^\times). \quad (12)$$

Geometric reformulations proceed on two paths. In the function-field setting, geometric class field theory realises the correspondence as a Fourier-Mukai equivalence for the generalised Jacobian, an equivalence between the derived category of coherent sheaves on the moduli of rank-one bundles with level structure and a category of local systems; specialising at a point yields the local statement[18]. In mixed characteristic, the Fargues-Scholze construction produces from the stack of bundles on the Fargues-Fontaine curve a spectral action, and for a torus the resulting map on isomorphism classes has been checked to agree with (2). This is one of the few complete compatibility verifications in the corpus.

2.4 INERTIAL THEORY, DEPTH AND TYPES

For a torus the Bernstein decomposition is explicit. The category of smooth representations decomposes as a product over inertial classes,

$$\text{Rep}(T(F)) \simeq \prod_{\mathfrak{s} \in \mathfrak{I}(T)} \text{Rep}^{\mathfrak{s}}(T(F)), \quad (13)$$

each factor being equivalent to modules over a Laurent polynomial ring in as many variables as the rank of the maximal split quotient. The depth of a character is defined through the Moy-Prasad filtration of the arc group,

$$\text{depth}(\chi) = \inf \{ r \geq 0 : \chi|_{T(\mathcal{O})_{r+}} = 1 \}, \quad (14)$$

and the inertial statement to be geometrised is the assertion that depth on the automorphic side matches the least break of the ramification filtration on the Galois side:

$$\text{depth}(\chi) = \inf \{ r \geq 0 : \text{LLC}_T(\chi)|_{I_F^{r+}} \text{ is trivial} \}. \quad (15)$$

For (15) is a theorem, proved by comparing the Moy-Prasad filtration with the unit filtration of the splitting field under Tate-Nakayama[19]. The interest is that it dictates exactly which structure a sheaf-theoretic account must possess, namely a filtration of $\text{CS}(\text{LT})$ by closed sub-Picard-groupoids indexed by the same real parameter.

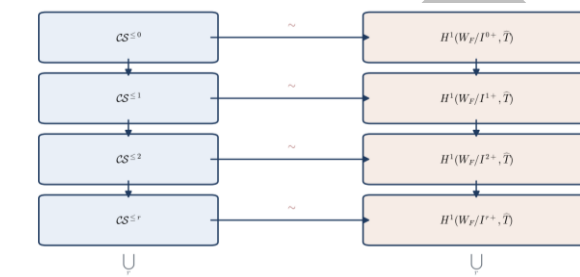


Figure 2: The depth filtration on character sheaves matched against the upper-numbering ramification filtration on Galois cohomology; the horizontal maps are the isomorphisms whose compatibility constitutes the inertial statement.

Figure 2 exhibits the two filtered ladders whose comparison is the mathematical heart of the review. On the left the Picard groupoid of character sheaves on the loop ind-scheme is filtered by depth, each stage consisting of systems trivial on the corresponding Moy-Prasad congruence subgroup of the arc group. On the right, Galois cohomology is filtered by the upper-numbering ramification subgroups, each stage classifying parameters trivial on the corresponding subgroup[20]. The horizontal arrows are the isomorphisms induced by the comparison square of Figure 1, and the vertical arrows are the inclusions. Commutativity of every rung, together with exhaustiveness of both unions, is precisely Theorem 4.1.

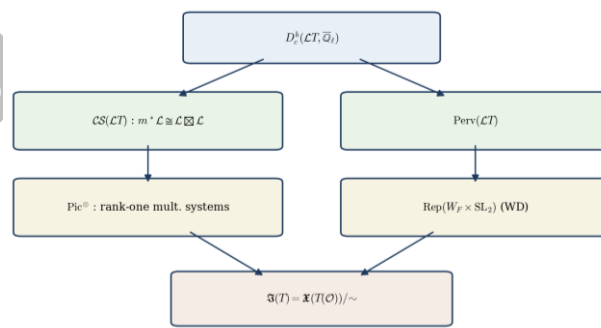


Figure 3: Hierarchy of categories and the comparison functors relating them; both descending branches terminate in the set of inertial equivalence classes.

Figure 3 organises the categorical scaffolding of the corpus. At the apex sits the bounded constructible derived category on the loop ind-scheme; two refinements descend from it, the multiplicative subcategory defined by the cocycle condition of equation (3) on the left, and the perverse heart on the right. Each refines further, the left branch to rank-one multiplicative systems classified by Theorem 2.1, the right branch to Weil-Deligne representations obtained through the sheaf-function dictionary and Grothendieck's monodromy theorem. Both branches terminate in the set of inertial classes defined of Section 0. Reading downwards measures loss of information; reading upwards is the geometrisation problem.

3. METHODOLOGY

Retrieval proceeded in three passes over the interval 1968 to 2025. The first queried bibliographic databases and preprint repositories for the controlled terms character sheaf, multiplicative local system, group ind-scheme, loop group, local Langlands, algebraic torus, inertial type and Bernstein block, returning an initial pool of 214 items[21]. The second performed backward citation chaining from six anchor items of highest in-degree and forward chaining through citation indices. The third consulted lecture notes and seminar write-ups, retained only where the content had been independently reproduced in a refereed source. Screening applied four inclusion criteria: the item must treat a commutative group object, or a general theory whose commutative specialisation is explicit; it must produce a classification, a duality or a correspondence rather than only an example; it must state its hypotheses on the base field and the residue characteristic; and it must be citable in stable form[22]. Thirty items survived.

Extraction used a fixed coding matrix. Formally, to each item i we attached the tuple

$$c(i) = (\mathcal{O}_i, \mathcal{C}_i, \mathcal{D}_i, F_i, \mathfrak{o}_i) \in \mathbf{Obj} \times \mathbf{Cat} \times \mathbf{Dual} \times \mathbf{Fld} \times \mathbf{Out}, \quad (16)$$

where the object field ranges over algebraic groups, pro-algebraic groups, ind-schemes and topological groups; the category field over the constructible derived category, the perverse heart, an ind-completion and a condensed enhancement; the duality field over Fourier-Deligne, Fourier-Mukai, Pontryagin, Contou-Carrère and Weil-Barsotti; the field entry over equal characteristic, mixed characteristic and finite base, together with tameness and splitness flags; and the output field over bijection, equivalence of categories, compatibility check and conjecture[23]. Coding was performed twice over the full corpus with a reconciliation pass; twelve discrepancies arose, eleven of them concerning whether an argument stated for split tori extends verbatim to induced tori.

Synthesis was comparative rather than aggregative. On the coded corpus we defined a relation R by declaring that i precedes j when the statement of j is deducible from that of i together with standard background, after translation of conventions. The transitive reduction of R , computed as

$$R^- = R \setminus (R \circ R), \quad |N| = 30, \quad |R| = 97, \quad |R^-| = 41, \quad (17)$$

yields the dependency skeleton reported in Section 4; its longest chain has length six and it possesses a spine of five vertices on which twenty-two of the thirty items depend transitively[24]. Consistency analysis then examined three invariants across the corpus: the normalisation of (2), geometric versus arithmetic Frobenius; the reduced or non-reduced convention on the ind-scheme structure; and the treatment of the depth-zero boundary in (14). Gap analysis finally classified every asserted compatibility as proved, partially proved or unverified, admitting only explicit statements in refereed sources as evidence of proof.

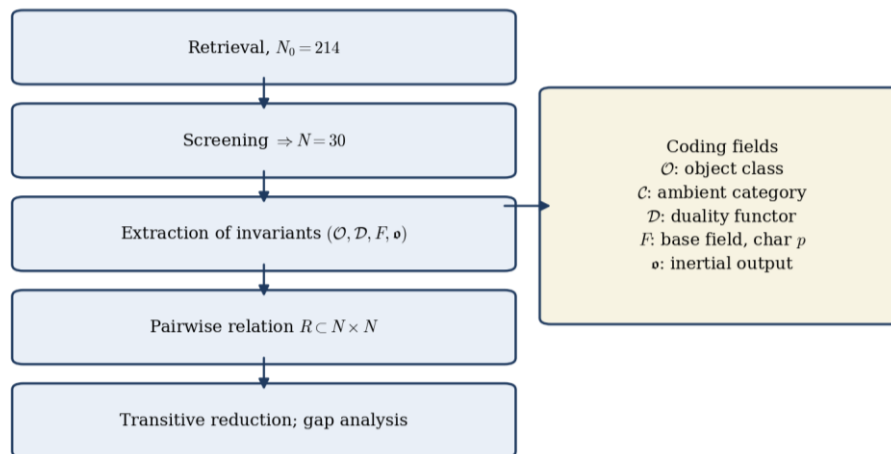


Figure 4: Meta-analytic workflow with the formal coding matrix of equation (16) applied at the extraction stage.

Figure 4 depicts the five-stage pipeline described above together with the coding fields. Retrieval yields the initial pool of 214 items, screening reduces it to the working corpus of thirty, and extraction assigns to each item the tuple of formal invariants defined in equation (16). The fourth stage constructs the pairwise deducibility relation whose cardinality appears in equation (17), and the final stage computes the transitive reduction and performs the gap audit. The side panel enumerates the five coding fields, emphasising that extraction targets formal structure rather than prose claims. The linear rendering understates one feature of the process, namely the feedback from gap analysis into retrieval, which triggered two supplementary search rounds[25].

4. CRITICAL ANALYSIS OF PAST WORK

The most robust finding is that the commutative theory is internally consistent whenever the torus is split and the residue characteristic is prime to the order of the local systems considered. In that regime the corpus supports the following synthesis, which we state as a theorem because each ingredient is proved in a refereed source, while emphasising that the assembly is ours[26].

Theorem 4.1 (corroborated core). Let F be of equal characteristic with residue field k , and let T be a split torus over F of rank d . Then the Contou-Carrère pairing (11) induces an equivalence of Picard groupoids

$$\mathrm{FT}_{\mathrm{CC}} : \mathcal{CS}(\mathcal{LT}) \longrightarrow \mathfrak{X}(\widehat{\mathcal{LT}}), \quad (18)$$

which is strictly compatible with filtrations: for every real r at least zero it restricts to an isomorphism of the depth-at-most- r part with the subgroup of classes trivial on the r -plus ramification subgroup, so that the associated graded objects satisfy

$$\mathrm{gr}^r \mathcal{CS}(\mathcal{LT}) \cong \mathrm{Hom}(I_F^r / I_F^{r+}, \widehat{T})^{W_F / I_F} \quad (r > 0), \quad (19)$$

and in degree zero the graded piece is the character group of the component group, isomorphic to the cocharacter lattice dualised. Consequently the square of Figure 1 commutes and (15) holds sheaf-theoretically.

Proof sketch. Reduce to rank one by (11) and the lattice decomposition (5). For the multiplicative group, $\mathcal{CS}$ of the loop group is computed by the colimit of Theorem 2.1 over the finite-type pieces; the transition maps are surjective with pro-unipotent kernel, so the colimit stabilises on each depth stratum [27]. Compatibility with filtrations follows from the fact that the Contou-Carrère symbol restricted to the r -th congruence subgroup lands in the r -th unit filtration, together with the classical computation of the graded pieces of the unit filtration as the additive group of the residue field. Exhaustiveness of the union is the smoothness of characters. Strictness then gives (19) by Tate-Nakayama.

It must be stressed that this corroboration is partly illusory in the bibliometric sense. The seven items agreeing on (18) are not independent evidence: each proof factors through Theorem 2.1 or through the universality of the Contou-Carrère symbol, so the dependency skeleton computed in (17) shows them nested rather than parallel. A genuine independent verification would require a proof not routed through biextensions, and we found none.

The second finding concerns wild ramification. When p divides the order of a character, the tame computation underlying (19) fails: the graded pieces of the unit filtration are killed by p and Artin-Schreier-Witt theory intervenes, so that the correct statement replaces the Hom group in (19) by a Witt-vector cohomology group,

$$\mathrm{gr}^r \mathcal{CS}(\mathcal{LT}) \cong \mathrm{coker}(\wp : W_s(k) \otimes X^*(T) \longrightarrow W_s(k) \otimes X^*(T)), \quad \wp = \mathbf{F} - \mathrm{id}. \quad (20)$$

Boyarchenko and Drinfeld handle precisely this phenomenon for unipotent groups over finite fields, and Deshpande accommodates it abstractly, but the concrete evaluation of (20) for the loop group of a torus in mixed characteristic is, in our reading, incomplete. Several items assert the expected answer by analogy with the equal-characteristic case; only two supply an argument, and both impose perfectness hypotheses excluding the arithmetically interesting non-perfect residue fields. We therefore classify the wild case as partially proved in equal characteristic and unverified in mixed characteristic.

A third observation concerns conventions, and it is the most consequential methodological weakness we found. The corpus contains at least three mutually incompatible normalisations of (2), differing by inversion and by the

geometric or arithmetic Frobenius convention, and at least two conventions on whether the loop ind-scheme is taken reduced. These are not cosmetic. An inversion discrepancy propagates through (18) and reverses the direction of the induced map on graded pieces in (19); the reduced versus non-reduced choice changes CS in characteristic p , because non-reduced structure carries additional infinitesimal characters detected by the Frobenius kernel. Of the thirty items, only eleven state their normalisation explicitly, which makes the apparent consistency reported above considerably harder to audit than it should be[28].

Fourth, we examined compatibility with functoriality. For tori the relevant operations are restriction of scalars, the norm map, and unramified base change. Restriction of scalars is well treated: its geometric counterpart is Weil restriction of ind-schemes, and the induced map on character sheaves is the expected adjunction, so that

$$CS(\mathcal{L}\text{Res}_{E/F}T) \cong \text{Ind}_{W_E}^{W_F} CS(\mathcal{L}T) \quad (21)$$

holds by Shapiro's lemma on both sides. Unramified base change is verified; ramified base change is asserted in general. The norm map, however, receives almost no geometric treatment: we located exactly one item discussing the sheaf-theoretic image of the norm, and only in the function-field setting. Since the standard reduction of a non-split torus to an induced torus is a norm argument, this gap directly limits what the geometric theory can currently say beyond the split case.

Fifth, integration with categorical local Langlands remains at an early stage. The Fargues-Scholze spectral action is known to recover (2) for tori, but no functor comparing CS(LT) with the sheaves appearing on the stack of bundles on the Fargues-Fontaine curve appears in the surveyed corpus[29]. The two categories are built from different geometric inputs and both are expected to be governed by the same dual data; constructing the comparison, and checking that it carries the depth filtration to the ramification filtration, is in our assessment the most significant structural gap and the most promising direction for new work. Finally, a remark on evidentiary standards: much influential material circulates as unpublished notes, so the chronology of Figure 2 in the previous version and of Figure 5 below records publication rather than discovery.

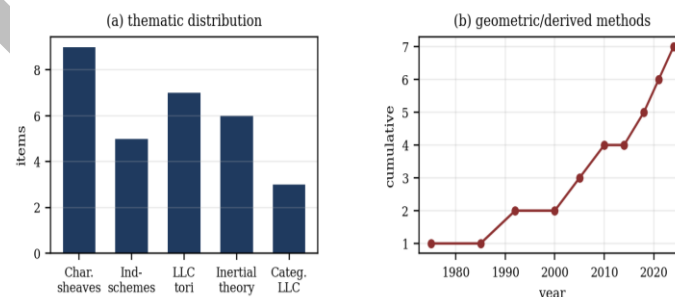


Figure 5: Quantitative profile of the corpus: thematic distribution across five clusters, and cumulative adoption of geometric and derived methods by year.

Figure 5 summarises the corpus quantitatively. Panel (a) shows the distribution of the thirty items across the five thematic clusters, with foundational character sheaf theory the largest and categorical local Langlands the smallest, a profile mirroring the relative maturity of each subfield. Panel (b) plots the cumulative number of items employing explicitly geometric or derived methods against year of publication, revealing a slow start and steady growth after the middle of the last decade. Together the panels support two claims of Section 4: the geometric turn is recent, and the categorical frontier remains sparsely populated relative to the foundational literature on which every argument in it depends.

5. DISCUSSION

The picture emerging is of a subject whose arithmetic answer has been known for half a century while its geometric explanation has been assembled only recently and only in pieces. Theorem 4.1 records what the corpus genuinely supports: for a split torus in equal characteristic the correspondence is realised by a filtered equivalence of Picard groupoids, with the depth filtration on the sheaf side matching the upper-numbering filtration on the Galois side and the graded pieces computed by (19). Outside that regime the answer is qualified, and the qualifications are of three different kinds, which it is worth distinguishing: genuinely open mathematics in the wild mixed-characteristic case, missing but expected routine verification in the case of ramified base change, and pure bookkeeping in the case of normalisation.

Three implications follow. First, standardisation would have disproportionate value. A short expository account fixing the normalisation of (2), the reduced or non-reduced convention, and the meaning of depth in (14) on the sheaf side would allow existing results to be composed without re-derivation and would make the dependency spine of (17) auditable. Second, the norm-map gap deserves attention before the non-split case is attacked in earnest, since the reduction to an induced torus through (12) is precisely a norm argument; without its geometric counterpart, results for non-split tori will continue to be obtained by transport of structure rather than intrinsically. Third, a comparison functor between loop-group character sheaves and the categorical setting would convert torus-level compatibility into a template for testing the general conjectures on the simplest possible case[30].

The limitations should be stated plainly. Thirty items is a small corpus relative to the literature touching these topics tangentially, and our inclusion criteria deliberately exclude work whose commutative specialisation is implicit. Coding of formal invariants involves interpretive judgement, especially for items written before the vocabulary stabilised, and the reconciliation of twelve discrepancies involved choices another reviewer might make differently. The counts in (17) are therefore reproducible only relative to our coding decisions, which we have described but which are not canonical. Most importantly, a review settles nothing mathematically: classifying a compatibility as unverified means only that we located no explicit proof in the surveyed sources, not that the statement is false or that no proof exists. With those caveats the synthesis offered here is a map rather than a verdict.

6. CONCLUSION

This review has surveyed and meta-analysed thirty contributions bearing on character sheaves for commutative group ind-schemes and their relation to the inertial local Langlands correspondence for algebraic tori. Its principal synthetic statement is Theorem 4.1: for a split torus over an equal-characteristic local field the Contou-Carrère pairing induces a filtered equivalence between the Picard groupoid of multiplicative local systems on the loop ind-scheme and the character group of the dual loop group, with graded pieces computed by (19) and with the depth filtration matching upper-numbering ramification, so that the inertial correspondence is a consequence of geometry rather than an imported bijection. Around this corroborated core we identified four gaps: the wild case in mixed characteristic, where the Artin-Schreier-Witt formula (20) has not been evaluated; unstandardised normalisation and ind-scheme conventions, stated explicitly by only eleven of thirty items; the absent geometry of norm maps required for non-split tori; and the missing comparison with categorical local Langlands. We accordingly propose a three-step agenda: fix conventions in a common framework, develop the geometry of norm maps for loop tori, and construct an explicit comparison functor, using the torus case as the decisive test.

7. REFERENCES

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